FORWARD MOVEMENTS IN SECONDARY MATHEMATICS

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In the *Bulletin of the American Mathematical Society* for December Professor David Eugene Smith devotes three pages to a review of a textbook bearing the title *First-Year Mathematics for Secondary Schools*. It would hardly be appropriate to attempt to answer a number of the criticisms that are made in this review if the only interest attaching to the review were the interest of the author of a single textbook. The fact that Professor Smith has devoted three pages in this *Bulletin* to the book is perhaps sufficient justification for regarding the review as one of some importance.

The book represents a concrete effort to embody in usable form a method of teaching mathematics which has been frequently discussed and frequently advocated in this country and abroad. Indeed, the book is a second edition of an experiment in combined algebra and geometry which is backed up by the long experience of the University High School of the University of Chicago.

Professor Smith's handling of this book shows that his interest in the volume is not essentially an interest in the particular form of combined mathematics which this book presents, but rather in the general problems of such a course. It will be appropriate, therefore, to comment on the points which he makes in the interest of the general movement for combined mathematics. The author of *First-Year Mathematics* may, therefore, be justified in setting aside the purely personal considerations which attach to his discussion, in his effort to call attention, in terms of those criticisms which Professor Smith presents, to the significance and value of the general movement.

In the first place, Professor Smith begins with a long statement which calls attention to the historical fact that "about six hundred fifty years ago Roger Bacon gave voice to his feelings with respect

to the teaching of mathematics, and this voice was in no respect uncertain nor was it at all lacking in emphasis." Professor Smith's statement goes on further to say that Bacon was in favor of combined mathematics. The reform which has been carried out in *First-Year Mathematics* is, therefore, according to Professor Smith's reasoning, a very ancient reform.

It would be interesting to draw parallel analogies in the history of civilization. I am told by my classical friends that Aristotle and Herodotus both suggested that some day men would probably fly in the air as birds do. I confess that I have none of the historical references which would make it possible to give the particular passages in which these comments are to be found. That flying is an ancient ambition of the human race would, however, seem to be fully proved by these references. The working out of the special mechanism by which flying can be effected has had to wait for the development of a good deal of modern machinery. The realization can probably be distinguished, by most historical students, from the first statement of the ambition.

It would be assuming too much to say that *First-Year Mathematics* is the only answer to Roger Bacon's prophecy. On the other hand, anyone who shuts his eyes to the fact that the movement for a combination of mathematics is vigorous, vital, and distinctly modern in its character is probably buried in historical details rather than watching to discern forward movements that are going on at the present time. Certainly history is interesting. It ought to be added, with equal emphasis perhaps, that recent efforts in Germany, England, and in this country to realize this great ambition of Roger Bacon and many a later mathematician are laudable in spite of the fact that the original discussion began six hundred fifty years ago, and has been blocked by the failure of educational machinery to supply the necessary instruments of the reform. Perhaps in this case, as in the world of mechanics, it is only recently that there have been sufficient flexibility and variety in modern life to carry the strain of an actual experiment in the realization of an ancient prophecy.

While touching on mathematical history, we may refer to a paragraph on the third page of Professor Smith's review. *First-*

Year Mathematics, following a number of other modern textbooks, has attempted to interest the student in the historical development of mathematical sciences and has included biographical sketches, distributed throughout the text, of a number of eminent students of mathematics. Professor Smith is very much grieved to find that in these the spelling of the names does not comport with his large historical information on the subject. The author of *First-Year Mathematics* must frankly admit that he borrowed the spelling of the names, together with most of the facts about these mathematicians, from such writers as Cajori, Ball, and others. Again, he must admit that he is not skilled in the critical study of mathematical history, and he will be very glad indeed to modify the spelling of these names; but he will be a little in doubt, even after the elaborate paragraph in Professor Smith's review, about how to attack the problem of getting the right spelling. If Professors Smith, Cajori, and Ball do not agree, obviously the ordinary student of mathematics must feel serious bewilderment in attempting to decide among them.

It may be well to comment in some detail on another general question which Professor Smith attacks in his review of this book. He says:

> Of course the book can be successfully taught; that is true of any book, provided the right teacher is available. But that a book with what seems to be a forced fusion of essentially different branches of a science, based solely upon the theory of ease of presentation, which theory does not seem to have been carried out—that such a book can be generally successful can hardly be expected.

The question which Professor Smith is discussing in this paragraph is one of vital interest to all teachers of mathematics. In his recent book on *The Psychology of High-School Subjects*, Professor Judd has brought together some statistics which seem to indicate that the problem of the teaching of mathematics is very much more involved than Professor Smith's remarks would seem to indicate. It will hardly be appropriate to print the full table which is given on p. 18 of the volume referred to. The failures in mathematics, as recorded, show a very much higher percentage than those in most of the other subjects of the curriculum. Students

are failing in Algebra I and II, in the schools reported, to the extent of 25 to 30 per cent. On the other hand, students are failing in English I and II only to the extent of 16 to 18 per cent. Students are failing in Latin I and II in numbers representing 18 to 25 per cent. In other words, the balance against mathematics, especially algebra, is so heavy as to make it clear that it is not possible for mathematics to hold its position in the secondary schools of this country unless the difficulty of presenting it to students is made a subject of careful study. It is the contention of the author of *First-Year Mathematics* that Algebra I, in the form in which it is ordinarily presented to students, is a badly organized subject. Indeed, *First-Year Mathematics* makes an effort to include a very large amount of geometry in the first-year course in high-school mathematics. That this is a distinct virtue is attested by the experience of the University High School and a number of other institutions that have used the book.

Rather than accept Professor Smith's general pronouncement that any book can be taught by a good teacher, we must admit on the basis of scientific studies that it is extraordinarily difficult, even for good teachers, to keep mathematics in the high school at the same level of success as the other subjects that are taught to students in these schools. That some reform in the teaching of mathematics must be worked out is therefore evident. The reform which is not uncommonly being made in high schools is the elimination of mathematics. Why one should take the position that every book serves the purposes of a good teacher is difficult to understand, when the whole effort of modern educational development is in the direction of a better organization of all the subjects of instruction. One might as well say that any textbook in Latin can be effectively used by a good teacher. The facts are, of course, that the good teacher is the teacher keenest in selecting that textbook which is well organized and well arranged for the needs of his pupils.

In exactly the same way it will hardly be possible very much longer to offer to teachers of mathematics the argument that they can use texts in mathematics of the conventional form, provided they are good teachers. The very fact that they are awake to the problems of instruction in the high schools will lead them to see the

necessity of making such a revision of the course in mathematics that it will be usable for the average student. The whole reform movement in mathematics has grown out of a very profound discontent with the kind of mathematics now taught in the schools. It is not serving its purpose, whether in the form of geometry or of algebra. Geometry is not needed by the ordinary student in the modern high school in the form in which it is now presented. It is very much needed in the form of constructive principles and devices. In the same way much of the algebra has absolutely no relation to anything except the most abstract intellectual life. Many of the sections on factoring, repeated again and again in the conventional textbook on algebra, are utterly without value to the students.

Mathematics can hardly go on in the beaten path which earlier generations have allowed it to take now that the course of study is filled with rich and concrete courses that bear upon life and give the student a stimulating view of the society and industry about him. When one realizes how easily it is possible to extract from algebra and geometry those principles which will be useful to the student, and to lay great stress upon the portions of mathematical science which are really available for intellectual development, it is hard to see why the conventional teacher tries to support the conventional book by saying that the book itself is of really very little value in determining the success of the course.

Stripping the review of Professor Smith of its historical subtleties and its sophistry with regard to instruction, we find a few paragraphs which have to do with the real problem of mathematics. Professor Smith objects violently to several sections in the book which he describes as formal. Fortunately he makes reference explicitly to pages, so that it is possible to know exactly what appeals to him as formal in the book; and one can, by comparative studies of other texts, including those which were written or revised by Professor Smith himself, set up contrasts which throw an interesting light on the general question of formalism in mathematics instruction.

The section on p. 5 to which Mr. Smith objects is the definition of equal segments, unequal segments, and notation for line-

segments. These exercises in the textbook are the results of a careful study of certain simple geometrical forms which are used as the concrete basis out of which the student may develop certain methods of expression. Geometry is used in the earlier pages of the book to establish, so far as possible, a concrete basis for the abstract discussions which begin on p. 5. The method of the book can thus be described by saying that it opens with concrete examples of certain mechanical principles and certain space relations, and gradually brings the student to the point where all of these can be expressed in a compact mathematical formula. The author is willing to accept Professor Smith's description, that p. 5 gives a formal statement of the conclusions of his discussions. All mathematics has the advantage of putting into a compact abstract form concrete relations which are much more complex when one views them in their concrete reality than they are when one expresses them in the abstract way. Formal statements of mathematical principles are certainly necessary in a mathematical discussion, but it does not follow that the treatment has been formal because it issues in a formal system of notation. Formalism in any proper sense of the word means a type of treatment which leaves the student with nothing except the bare husks or outlines of the idea. A lesson is formal when it rushes too soon into the sphere of abstraction. A lesson is unjustifiably formal when it gives the student no content about which he may think.

First-Year Mathematics does not at any point lay claim to an abandonment of the formal side of mathematics, but it does make an effort in every case to lead the student step by step, with great detail of concrete illustration, to an intelligent interpretation of the formal symbols which are employed. As opposed to the method of ordinary books in mathematics, this effort to be concrete is one of the major characteristics of combined mathematics.

The author of *First-Year Mathematics* finds great satisfaction, as he reads the modern books in algebra, in the more liberal use made by other authors of the graph and other geometrical devices for illustrating abstract number relations. That even those who are unwilling to go the whole length of combining the two subjects are thus constantly employing methods that are virtually those

of combined mathematics seems to him to be a clear indication that a further development of this attitude will not be unwholesome for the science itself, and will ultimately free mathematics from that objectionable formalism which he would join Professor Smith in condemning in unqualified terms.

By way of further justification of the method employed in *First-Year Mathematics*, it is legitimate to point out that many authors of textbooks on secondary mathematics assume that the fundamental formal concepts of mathematics are so simple that the average student can understand them, even when they are stated without any concrete illustrations whatsoever. The result is that students begin work in mathematics with vague notions. Years may pass before they understand them clearly, and then this belated understanding is brought about usually by interpreting the abstract statement in terms of some concrete illustration. The author of *First-Year Mathematics* has aimed to provide content wherever the student meets a definition or an axiom for the first time.

A few typical examples will illustrate the difference between the undesirable formal treatment of algebra or geometry and the use of mathematical forms as these are taken up in combined mathematics.

First, we may restate briefly the method used in *First-Year Mathematics* in taking up the axioms:

The teacher draws on the blackboard two pairs of equal line-segments (Fig. 1), as AB and CD, and EF and GH. A student is then asked to draw on the board the sums $AB+EF$ and $CD+GH$. Another student measures these sums and compares them. He finds that they are equal.

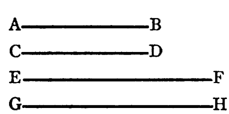

FIG. 1

The problem is then repeated in a somewhat different form: Letting a, b, c, and d be the lengths of four segments such that $a=b$, and $c=d$, show by measuring that $a+c=b+d$. By means of these problems the student is led to state the addition axiom in his own words. He finds in the textbook, on the other hand, a statement of the axiom with which to compare his own: *Equals added to equals give equals.*[1] The second problem makes him familiar,

[1] *First-Year Mathematics*, p. 17.

while he is learning the statement, with the form in which the axiom is usually applied. At the same time he is acquiring valuable training in the ability to measure—so important for all graphical work. The subtraction axiom is approached two pages farther on in a similar manner (p. 19). Discussion of the multiplication axiom is postponed until p. 35; and the division axiom appears on p. 34. No attempt is made to hurry the student over several of the axioms at one time. Clear comprehension, not quantity, is the principal purpose of this beginning work. Each axiom is followed by a variety of problems giving opportunity to apply the new principle.

Presented thus, the study of algebra and geometry may be taken up earlier than is now commonly the case. The writer has a seventh-grade class studying *First-Year Mathematics* and so far has found no difficulty in getting the pupils to do the work satisfactorily. At the end of the first semester these twenty-one children took the same final examination as the regular first-year classes in high school, and their grades compared favorably with those of the high-school students.

Turning to the treatment of the axioms in the *Wentworth-Smith Algebra*, p. 20, we find that not one but six axioms are given at one time. The textbook indicates no connection between these abstract statements and what precedes. It is evidently left to the student or to the good teacher who can use any book to make this connection. The axioms are stated as follows:

> *Axioms:* A general statement admitted to be true without proof is called an *axiom*. The axioms in the beginning of algebra are six in number.
> 1. If equals are added to equals, the results are equal.
> 2. If equals are subtracted from equals, the results are equal.
> 3. If equals are multiplied by equals, the results are equal.
> 4. If equals are divided by equals, the results are equal.
> 5. Like powers, or like roots, of equals are equal.
> 6. Quantities equal to the same quantity are equal to each other.

Without further explanation, this is followed by problems of this type: "If $2a=16$, what does a equal? What axiom is used?"

Those of us who have worked sympathetically with high-school students have found that, to the average student who has been in the high school a little over a week, an axiom has very little meaning when approached as above. How difficult must it be for him to select and apply the proper axiom to be used in these problems! Sooner or later he will suffer the fatal consequences of the

error of assuming that these things are simple. The later work of the year and of subsequent years will show that as tools for problem-solving these axioms are of little value to the student, because he has neither understood nor applied them.

Of course, the teacher will find that he can get an answer promptly: "If $2a=16$, then $a=8$." But that does not at all indicate that the student has used an axiom to get the result. In all probability his former training in arithmetic helps him to get the answer by inspection. This is very different from using the axiom, when he should reason about as follows:

Given	$2a=16$
Dividing both sides of the equation by 2,	$\frac{2a}{2}=\frac{16}{2}$
Reducing to its simplest form,	$a=8$

If the student learns how to work the example but does not understand what he does, all may go well until he tries to solve an equation like $ax=b$. Unless he is conscious of the fact that he must divide both sides by a, he is as likely to answer $x=b-a$, as $x=\frac{b}{a}$. Once more he is made to go back to the equation $2a=16$. This time he reasons out with great care why the axiom gives him $a=8$, so that he may be able to pass to the more difficult step $x=\frac{b}{a}$, and, probably for the first time during the course he understands what the division axiom really means.

Many students do not acquire the power to use this axiom in the second or even the third year of the high-school course. A common illustration of this fact is found in assigning problems like the following: "Prove that if a line bisects two sides of a triangle it is parallel to the third side." The suggestion is given that this will be true if it can be shown that the segments of the first two sides are proportional. In symbols, the problem may be stated thus: "Given $a=b$, $c=d$ (Fig. 2); prove that $\frac{a}{c}=\frac{b}{d}$." We see that the desired proportion is obtained by applying the division axiom. Yet, after a year's work in algebra, many students are unable to draw this apparently simple conclusion.

The writer has seen the same difficulty arise even later. In a third-year class a certain problem had been worked up to the point where the following equations were obtained:

$$450=c(x-4)$$
$$425=c(y+3)$$

The required unknowns are x and y. The number c is to be eliminated. After some discussion of the method of doing this, the class decided to eliminate c by solving each equation for c and then comparing the results. Only one member of that class was able to see that he might eliminate c by using the division axiom.

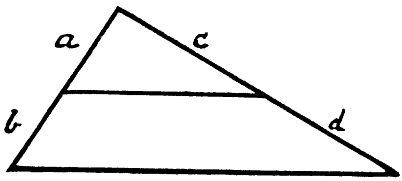

FIG. 2

In the light of much experience would it not be better to remove the vagueness and subsequent uncertainty by making each law clear at the beginning, rather than to assume that the ordinary student can comprehend six of the axioms at once? It is hard to understand why it is that some authors regard many difficult matters of elementary mathematics as simple. For the high-school teacher it is a far safer course to follow the principle "never to underestimate the difficulty of understanding on the part of the pupil" than to dispose too quickly of very important, abstract, and fundamental laws as "simple." Regarding the effort of the author of *First-Year Mathematics* to illustrate some of these so-called simple things in a concrete way, Professor Smith disapprovingly says: "The reader will find the simple made difficult in various cases, as in such products as $(a-b)c$, $(a+b)(a-b)$, $(a-b)(c+d)$, and $(a-b)^2$, and in the law of signs in multiplication as based upon the 'turning tendency' idea." Experience has shown that some of the products mentioned here as well as the law of signs in multiplication appear not at all simple to beginners. One may visit classes in mathematics in almost any high school and be sure to find students who will state that $(a-b)^2$ is equal to a^2-b^2. Indeed, some students find the correct form, $(a-b)^2=a^2-2ab+b^2$, one of the hardest things in algebra to remember. The difficulty occurs in the first year as well as in the second or third, if not in the fourth. Even college professors of mathematics are heard to

complain that the principle did not receive in the high school the attention which it deserves. Surely no experienced high-school teacher can call this a simple matter. Nor is this difficulty typical of the present high-school generation. From his own high-school days the writer remembers only too well that students had the same trouble in squaring binomials.

If concrete illustrations help some students to remember the correct expansions of such expressions as $a(a+b)$, $(a+b)^2$,$(a-b)^2$, etc., as experience has shown that they do, why should they not be given in a textbook?

Since the law of signs in multiplication is mentioned as one of the simple things of algebra, the reader will be interested in Professor Smith's own treatment of this law with comments interspersed as follows:[1]

We cannot pick up a book $2\frac{2}{3}$ times. [Some high-school students will not be willing to admit that.] Nevertheless, we say that $2\frac{2}{3}$ times \$3 equals \$8. [Will the student see the connection between these two statements, and if he does, will that type of reasoning appeal to him as being clear?] That is, we define [This is expecting much of a student who entered high school about three weeks before. If we must take something as a definition, why not take the sign law itself?] what is meant by multiplying by $2\frac{2}{3}$ and then use the word "times" just as we do with integers. Similarly [i.e., we are to assume the statement above to be perfectly clear], we cannot pick up a book -2 times, but we may define what we mean by multiplying by -2, and then we may use the word "times" as we do with positive numbers. [This will hardly seem simple to a beginner in algebra.] Because $3\times(-2)=-6$, therefore $(-2)\times 3$ ought to equal -6. [Notice that the commutative law is used here. In his review Professor Smith objects to the early introduction of this law. Should we then use the principle without stating or naming it? We do not hesitate to use such terms as coefficient, exponent, polynomial, etc. What objection, then, is there to the term "commutative"?] Therefore, we define multiplication by a negative number to mean multiplication by a positive number having the same absolute value, the sign of the product being changed. [Even an adult will find it difficult to follow the author through this kind of reasoning.]

Therefore, $2\times(-3)=-6 \qquad a(-b)=-ab$

$-2\times 3=-6 \qquad -a\times b=-ab$

$-2\times(-3)=6 \qquad -a\times(-b)=ab$

If two numbers have like signs, their product is positive.
If two numbers have unlike signs, their product is negative.
[This concludes the treatment.]

[1] *Wentworth-Smith Algebra.* Book I, § 32. The remarks in brackets are questions raised by the writer.

Even if a student has succeeded in following the preceding discussion, he will be quite willing to admit that the law itself is the simplest part of it. However, on p. 60 this is made still simpler. Here we find the statement: "Briefly stated, the law of signs in multiplication is as follows: In multiplication two like signs give *plus;* two unlike signs produce *minus.*" Unfortunately there is danger that some students continue the process of simplifying the law. They omit the "multiplication," and simply remember that "like signs give plus, unlike signs, minus." During the remainder of their high-school careers, they continue to apply this wherever they meet like signs. For example, they will assert that $(-2)+(-3)=+5$. How are we now to convince them that they are wrong? Will it help them to go a second time through the mental performance described above?

We should recognize, in the first place, that the laws of signs are *not simple* for beginners. For this reason they should not be discussed until the student has had considerable experience in the operations with literal numbers. In *First-Year Mathematics* the law of signs in multiplication is taken up after the student has had a half-year's work in high-school mathematics. Nor is it based on the "turning tendency" idea. It is approached by means of four problems, as follows:

1. Find the product of $(+4)$ by $(+3)$.

Solution: Since $(+3)(+4)$ is the same as $(3)(+4)$, it follows that $(+3)(+4)$ equals $(+4)+(+4)+(+4)=(+12)$. Geometrically this means that to multiply $(+4)$ by $(+3)$ is to lay off $(+4)$ three times in its *own* direction (Fig. 3). Thus, $(+3)(+4)=(+12)$

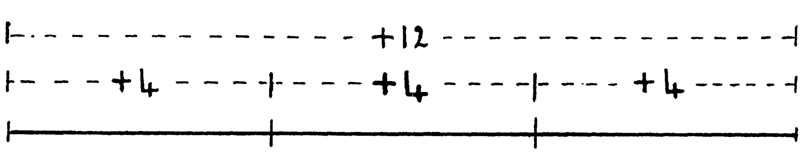

FIG. 3

2. Find the product of (-4) by $(+3)$.

Solution: Since $(+3)(-4)$ is the same as $(3)(-4)$, $(+3)(-4)=(-4)+(-4)+(-4)=(-12)$. Make a drawing for $(+3)(-4)$, i.e., lay off (-4) three times in its *own* direction. Thus, $(+3)(-4)=(-12)$.

3. Find the product of $(+4)$ by (-3).

Solution: Assuming that the commutative law holds for positive and negative numbers [the student is familiar with this law] it follows that $(-3)(+4) = (+4)(-3) = (-12)$. Notice that the same result is obtained by laying off $(+4)$ three times in the direction *opposite* to its own direction. Make a drawing for this product.

4. Find the product of (-4) by (-3).

According to problem 3, this means that (-4) is to be laid off three times in the direction opposite to that of (-4). Thus, $(-3)(-4) = (+12)$.

Then, from a study of the results of problems 1 to 4, the law of signs is deduced.

Having made clear this law by the use of segments, the most concrete material with which the student is familiar, it is illustrated once more, this time with the lever. But this is not the main purpose of the introduction of the lever at this point. Acquaintance with it opens to the student a whole field of problems in which he is greatly interested and which may be solved by means of algebraic equations (see *First-Year Mathematics*, p. 273). Furthermore, it gives him a clear understanding of the directions of turning and of positive and negative angles needed later in the study of geometry and trigonometry. Moreover, to him the lever is as interesting a piece of apparatus as it is simple. It is easy for a teacher of mathematics to secure a lever from the physics department, or if necessary to make the apparatus. What, then, can be the objection to this additional illustration of the law of signs in multiplication? Professor Smith does not object to the use of notions of physics, as is shown by his own method of illustrating the law of signs in addition (see *Wentworth-Smith Algebra*, p. 28). Or are we to assume that this law is more difficult than that of multiplication? Hardly. Then why use notions of physics in one and object to them in the other? In fact, a student can, and many students do, get along altogether without the law of signs in addition. Thus, when adding (-1) to $(+10)$, he may reason as follows: A loss of 1 followed by a gain of 10 is a gain of 9. In a similar way he may settle every question of signs in addition.

Now let us see how successfully Professor Smith uses physics as a means of making concrete the law of signs in addition. He discusses this law as follows:

If we tie to a 10-pound weight a toy balloon that pulls upward 1 pound, what will the two together weigh? [As a problem *following* the discussion of

the sign law this would not be objectionable. But does it serve its purpose as a means of introduction and interpretation? The student will probably wonder how a balloon could be weighed. The idea is new. He has never had occasion to purchase balloons by weight.] From the answer to the question above we find[?] that "to add a positive number to a negative number, take the difference of their absolute values and prefix the sign of the numerically greater number. Similarly, to add a negative number to a negative number take the sum of their absolute values and prefix the negative sign."

Notice that all this is to come out of the *answer* to *one* problem which was most likely given by some bright pupil in the class.

But here we should point to the accompanying picture (Fig. 4). Will this clear up the situation? The student who has been deprived of natural and illuminating training in geometry will probably see in this picture nothing more than a circle attached to something that looks like a post card. How can that picture give him the necessary ideas as to pull and motion, as in the case of lever and weights? If we could make analyses of what is going on in the minds of all students of an ordinary first-year class when trying to follow the discussion above, all the difficulties just mentioned would appear. The boy never saw a balloon pulling upward a pound. The experiment cannot be performed in the class room, since the teacher is unable to secure a balloon that will pull upward a pound. Why is the upward direction of this balloon negative when it is usually considered positive? These and other questions will puzzle the student as he is trying to master a law which would be exceedingly simple if line-segments were used, or the gain-and-loss illustration given above.

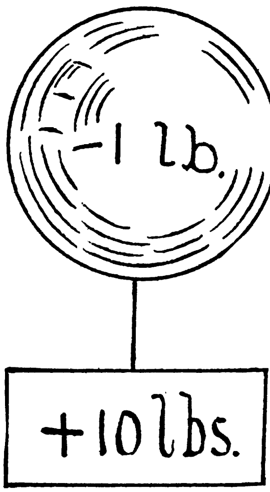

FIG. 4

A systematic comparison of a textbook of the old type with a representative text of the new type would throw considerable light on the great difficulties in high-school mathematics, and would help to explain why more students fail in mathematics than in

other subjects. It would also help to answer some questions raised by Professor Smith. "Given the average teacher," he says, "will the student at the end of his work in the high school be as well grounded in mathematics as he would be if the work had been arranged on some other plan?" That depends, of course, entirely on what the other plan is. If by the other plan is meant the old type of method in mathematics, it would be possible to furnish statistics from the University of Chicago High School and from other schools that justify fully the answer yes. "Will he appreciate the subject as well or be as likely to continue his study of its higher branches?" This question may be answered like the preceding one.

Finally he asks: "Does the book (*First-Year Mathematics*) meet the ideals which he [the author] himself has laid down?" The writer feels that he has attempted a solution of a vital and most difficult problem. He thinks that considerable progress has been made in the right direction, but he is far from believing that the task is completed. However, an exceedingly friendly and growing group of teachers is using the book. The author is keeping in close touch with them, is carefully studying their difficulties whenever they arise, and intends to keep on improving the book until his ideals are realized.